Solutions to some triple integral problems

For each of the following, set up an iterated integral equal to the given triple integral.

1. $\iiint\limits_R (x + 2y - z)\, dV$ where $R = [0, 2] \times [-4, 6] \times [-3, 0]$

   Solution: $\iiint\limits_R (x + 2y - z)\, dV = \int_{-3}^{0} \int_{-4}^{6} \int_{0}^{2} (x + 2y - z)\, dx\, dy\, dz$

2. $\iiint\limits_R (x + 2y - z)\, dV$ where R is the solid region bounded between the graph of $z = x^2 + y^2$ and the plane $3x + 5y + 2z - 12 = 0$

   Solution: The graph of $z = z^2 + y^2$ is a paraboloid. We find the intersection between the paraboloid and the plane by equating z for each to get

$$x^2 + y^2 = -\frac{3}{2}x - \frac{5}{2}y + 6.$$

By completing the square on the x terms and on the y terms, we can rewrite this as

$$(x + \frac{3}{4})^2 + (y + \frac{5}{4})^2 = \frac{65}{8}.$$

This is the equation of a circle of radius $r_0 = \sqrt{65/8}$ centered at the point $(-3/4, -5/4)$. This circle is the projection of the intersection between the paraboloid and the plane. For points of the xy-plane inside the circle, the solid region extends in the z-direction from $z = x^2 + y^2$ to $z = -\frac{3}{2}x - \frac{5}{2}y + 6$. To describe the points of the xy-plane inside the disk, let's introduce some temporary new coordinates $u = x + \frac{3}{4}$ and $v = y + \frac{5}{4}$. In uv-coordinates, the equation of the circle is

$$u^2 + v^2 = r_0^2.$$

We can choose constant bounds for u to get

$$-r_0 \le u \le r_0 \qquad \text{and} \qquad -\sqrt{r_0^2 - u^2} \le v \le \sqrt{r_0^2 - u^2}$$

as our description of the disk. Translating back to x and y gives

$$-r_0 \le x + \tfrac{3}{4} \le r_0 \qquad \text{and} \qquad -\sqrt{r_0^2 - (x + \tfrac{3}{4})^2} \le y + \tfrac{5}{4} \le \sqrt{r_0^2 - (x + \tfrac{3}{4})^2}$$

so we can describe the disk by

$$-r_0 - \tfrac{3}{4} \le x \le r_0 - \tfrac{3}{4} \qquad \text{and} \qquad -\sqrt{r_0^2 - (x + \tfrac{3}{4})^2} - \tfrac{5}{4} \le y \le \sqrt{r_0^2 - (x + \tfrac{3}{4})^2} - \tfrac{5}{4}.$$

A complete description of the solid region is thus given by

$$\begin{aligned} -r_0 - \tfrac{3}{4} \le & x \le r_0 - \tfrac{3}{4} \\ -\sqrt{r_0^2 - (x + \tfrac{3}{4})^2} - \tfrac{5}{4} \le & y \le \sqrt{r_0^2 - (x + \tfrac{3}{4})^2} - \tfrac{5}{4} \\ x^2 + y^2 \le & z \le -\tfrac{3}{2}x - \tfrac{5}{2}y + 6 \end{aligned}$$

We can thus write

$$\iiint\limits_R (x+2y-z)\,dV = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_2(x,y)} (x+2y-z)\,dz\,dy\,dx$$

where the limits of integration can be read off from the description of the solid region.

Using *Mathematica*, I evaluated this iterated integral and got

$$\iiint\limits_R (x+2y-z)\,dV = \frac{-1094275\,\pi}{3072} \approx -1119.06.$$

3. $\iiint\limits_R 1\,dV$ where R is the solid region bounded by the surface $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$, the plane $z=-1$ and the plane $z=2$

Solution: The graph of $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$ is a hyperboloid of one sheet with main axis along the z-axis and elliptic cross-sections parallel to the xy-plane. We are given constant bounds on z so consider the yz cross-section. (One could also choose to work with the xy cross-section here.) In the yz-plane, the relevant region is between the two branches of the hyperbola $\frac{y^2}{9}-z^2=1$ from $z=-1$ to $z=2$. We can describe this planar region by

$$-1\le z\le 2 \qquad \text{and} \qquad -3\sqrt{1+z^2}\le y\le 3\sqrt{1+z^2}.$$

For each point of the yz-plane in this region, the solid region extends in the x direction from one side of the hyperboloid to the opposite side. The relevant bound on x come from solving $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$ for x. A complete description of the region is thus

$$\begin{aligned} -1 &\le z \le 2 \\ -3\sqrt{1+z^2} &\le y \le 3\sqrt{1+z^2} \\ -2\sqrt{1+z^2-\tfrac{y^2}{9}} &\le x \le 2\sqrt{1+z^2-\tfrac{y^2}{9}} \end{aligned}$$

Thus,

$$\iiint\limits_R 1\,dV = \int_{-1}^{2}\int_{y_1(z)}^{y_2(z)}\int_{x_1(y,z)}^{x_2(y,z)} dx\,dy\,dz$$

where the limits of integration can be read off from the description of the solid region.

Using *Mathematica*, I evaluated this iterated integral and got

$$\iiint\limits_R 1\,dV = 9\left(\sqrt{2}+2\sqrt{5}+\operatorname{arcsinh}(1)+\operatorname{arcsinh}(2)\right)\approx 73.9.$$

We can interpret this result as the volume of the solid region because the integrand is 1. For comparison, note that this solid region fits inside a cylinder of radius 3 and height 3 for which the volume is $\pi(3)^2(3)\approx 84.8$.

Note: The function arcsinh is the *inverse hyperbolic sine function.*